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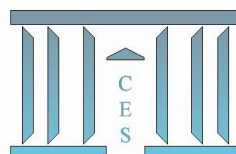
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**On the continuous representation of quasi-concave
mappings by their upper level sets**

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On the continuous representation of quasi-concave mappings by their upper level sets.

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Abstract: we provide a continuous representation of quasi-concave mappings by their upper level sets. A possible motivation is the extension to quasi-concave mappings of a result by Ulam and Hyers, which states that every approximately convex mapping can be approximated by a convex mapping.

Keyword: quasi-concave, upper level set

AMS classification: 47N10

1 Introduction

Since a seminal paper of Hyers and Ulam (1), several papers have studied the problem of the stability of functional inequalities. In particular, the first issue given (and solved) by Hyers and Ulam was roughly the following: is it possible to approximate approximately convex mappings by convex mappings?

Now, an important extension of convexity, especially for mathematical economics or game theory, is quasi-concavity. In the following, let X be a convex subset of some vector space.

Definition 1.1 *A mapping $f : X \rightarrow \mathbb{R}$ is quasi-concave if the following*

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property is true:

$$\forall (x, y) \in X \times X, \forall \lambda \in [0, 1], f(\lambda.x + (1 - \lambda)y) \geq \min\{f(x), f(y)\}.$$

Equivalently, $f : X \rightarrow \mathbb{R}$ is quasi-concave if its upper level sets $S_\lambda(f) = \{x \in X, f(x) \geq \lambda\}$ are convex subsets of X for every $\lambda \in \mathbb{R}$. Clearly, concave and convex mappings are quasi-concave. Besides, non-increasing or non-decreasing mappings from \mathbb{R} to \mathbb{R} are quasi-concave.

In order to raise the issue of the stability of the quasi-concavity property, one defines the notion of approximately quasi-concave mapping as follows:

Definition 1.2 *For every $\epsilon > 0$, the mapping $f : X \rightarrow \mathbb{R}$ is ϵ -quasi-concave if the following property is true:*

$$\forall (x, y) \in X \times X, \forall \lambda \in [0, 1], f(\lambda.x + (1 - \lambda)y) \geq \min\{f(x), f(y)\} - \epsilon.$$

A natural question, in the vein of Hyers and Ulam's stability result, is to know whether it possible to approximate an ϵ -quasi-concave mapping $f : X \rightarrow \mathbb{R}$ by a quasi-concave mapping. Such question could have applications in game theory or in mathematical economics, but it also has an interest for its own sake.

A possible answer to this question could be to build the quasi-concave envelop \tilde{f} of f as follows: first, associate to the ϵ -quasi-concave mapping f its upper level sets $S_\lambda(f) = \{x \in X, f(x) \geq \lambda\}$.¹ Then convexify the upper level sets by defining their convex hulls $\text{co}S_\lambda(f)$. Last, try to find a quasi-concave mapping \tilde{f} whose upper level sets are the $\text{co}S_\lambda(f)$.

A first objective of the present paper is to clarify the previous construction.² In particular, one could hope that the mapping \tilde{f} is a good approximation of f if ϵ is small enough. In view of the construction of \tilde{f} , a question which naturally arises is the continuity of the representation of each mapping $f : X \rightarrow \mathbb{R}$ by its upper level sets. This is the main purpose of this paper, and this study is given in Section 2. In particular, it requires the definition of a good topology on the set of upper level sets. Besides, the stability problem

¹The representation of a mapping by its upper level sets is a well nown process (see, for example, Crouzeix (2), chapter 3.) In this paper, we are more interested in the continuity of such representation.

²Other constructions of the quasi-concave envelop could be given. But we think the construction here given, which is very natural, raises interesting questions we would like to answer in this paper.

presented above suggests to study the continuity properties of the convex hull operator in the set of upper level sets, which is done in Section 3. The stability of the quasi-concave property is given in Section 4, as a immediate byproduct of the previous sections.

2 Continuous representation of a quasi-concave mapping by its upper level sets

The purpose of this section is to prove that the standard representation of quasi-concave mappings by their upper level sets can be made continuous by defining adapted topologies.

In the following subsection, we recall the standard representation result.

2.1 Definition of u.l.s. multivalued mappings and the standard representation of a mapping by its upper level sets.

Definition 2.1 *Let X be a vector space. A multivalued mapping T from \mathbb{R} to X is said to be a u.l.s. (upper level sets) multivalued mapping if the following properties are satisfied:*

- i) **Monotony** *For every $(\lambda, \mu) \in \mathbb{R} \times \mathbb{R}$, $\lambda \leq \mu$ implies $T_\mu \subset T_\lambda$.*
- ii) **Left Continuity** *For every $\lambda \in \mathbb{R}$, one has $\bigcap_{n \in \mathbb{N}^*} T_{\lambda - \frac{1}{n}} = T_\lambda$.*
- iii) **Limits at $+\infty$ and $-\infty$** *One has $\bigcup_{\lambda \in \mathbb{R}} T_\lambda = X$ and $\bigcap_{\lambda \in \mathbb{R}} T_\lambda = \emptyset$.*

Throughout this paper, \mathcal{T} denotes the set of all multivalued mappings from \mathbb{R} to X satisfying the Assumptions i) and iii) above, and $\mathcal{T}^{u.l.s.}$ the set of all u.l.s. multivalued mappings from \mathbb{R} to X . Besides, \mathcal{F} denotes the set of all mappings from X to \mathbb{R} .

We recall the following well known result:

Theorem 2.2 *(See, for example, Crouzeix (2), chapter 3.)*

The mapping $\Phi : \mathcal{F} \rightarrow \mathcal{T}^{u.l.s.}$ defined by

$$\forall f \in \mathcal{F}, \forall \lambda \in \mathbb{R}, \Phi(f)(\lambda) = S_\lambda(f) := \{x \in X, f(x) \geq \lambda\}$$

is bijective and $\Psi = \Phi^{-1}$ is defined by

$$\forall T \in \mathcal{T}^{u.l.s.}, \Psi(T)(x) = \sup\{\lambda \in \mathbb{R} \mid x \in T_\lambda\}.$$

Remark 2.3 Condition iii) of Definition 2.1 guarantees that f takes finite values. In (2) is given a version of Theorem 2.2 for quasiconvex mappings whose values can be infinite, so that Condition iii) is not useful. The case of Theorem 2.2 here given is straightforward.

Now, we would like to refine the previous representation result, in order to obtain some continuity of this representation.

2.2 Definition of a topology on \mathbf{T}

In the following Proposition 2.4, whose proof is given in the appendix, we define a generalized metric structure on $\mathcal{T}^{u.l.s.}$. Recall that for every set E , a mapping $\delta : E \times E \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ is a generalized pseudo metric if these three conditions are true:

- (i) for every $x \in E$, $d(x, x) = 0$.
- (ii) for every $(x, y) \in E \times E$, $d(x, y) = d(y, x)$.
- (iii) for every $(x, y, z) \in E^3$, one has $d(x, z) \leq d(x, y) + d(y, z)$.

The generalized pseudo metric d is a generalized metric if in addition, for every $(x, y) \in E^2$, $d(x, y) = 0$ implies $x = y$.

In the following proposition, by convention, the infimum of an empty set is $+\infty$.

Proposition 2.4 The mapping $\delta : \mathcal{T} \times \mathcal{T} \rightarrow \mathbb{R}^+ \cup \{+\infty\}$ defined for every $(T, T') \in \mathcal{T} \times \mathcal{T}$ by

$$\delta(T, T') = \sup_{\lambda \in \mathbb{R}} \{\max\{\inf\{a \geq 0 \mid T_\lambda \subset T'_{\lambda-a}\}, \inf\{a \geq 0 \mid T'_\lambda \subset T_{\lambda-a}\}\}\}$$

is a generalized pseudo-metric, and its restriction to $\mathcal{T}^{u.l.s.} \times \mathcal{T}^{u.l.s.}$ is a generalized metric.

Remark 2.5 It is easy to build a metric from the generalized metric δ , for example by defining $d : \mathcal{T}^{u.l.s.} \times \mathcal{T}^{u.l.s.} \rightarrow \mathbb{R}^+$ by

$$\forall (T, T') \in \mathcal{T}^{u.l.s.} \times \mathcal{T}^{u.l.s.}, d(T, T') = \min\{1, \delta(T, T')\}.$$

We now define a topology on \mathcal{T} , thanks to the generalized pseudo metric δ defined above.

Definition 2.6 *Let Top be the topology on \mathcal{T} induced by the previous generalized pseudo metric δ . More precisely, this topology is generated by the collection of open sets $B(x, r) = \{y \in \mathcal{T}, \delta(x, y) < r\}$ where $x \in \mathcal{T}$ and $r > 0$.*

Throughout this paper, \mathcal{T} is equipped with the topology Top , and the topological space such defined will be denoted (\mathcal{T}, δ) . Notice that from the properties satisfied by the generalized pseudo-metric δ , the collection of sets $B(x, r)$ is a basis of Top : it means that every open set for Top is a union of some $B(x, r)$.

Since we aim at representing continuously quasi-concave mappings from X to \mathbb{R} by elements of $\mathcal{T}^{u.l.s.}$, we now have to define a topology on \mathcal{F} .

Definition 2.7 *For every $(f, g) \in \mathcal{F}$, define $\|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|$. Let Top' be the topology on \mathcal{F} generated by the collection of open sets $B(f, r) = \{g \in \mathcal{F}, \|f - g\|_\infty < r\}$, where $f \in \mathcal{F}$ and $r > 0$.*

Remark that $\|\cdot\|_\infty$ is not a norm. Nevertheless, $(\mathcal{F}, \|\cdot\|_\infty)$ will denote the set \mathcal{F} equipped with the topology \mathcal{U} . Notice that the collection of sets $B(x, r)$ is a basis of Top' .

2.3 The continuous representation

Let \mathcal{F}^q be the set of all quasi-concave mappings in \mathcal{F} , and for every $\epsilon > 0$, let \mathcal{F}_ϵ^q be the set of all ϵ -quasi-concave mappings in \mathcal{F} . The sets corresponding to \mathcal{F}^q and \mathcal{F}_ϵ^q through our representation will be \mathcal{T}^{conv} and $\mathcal{T}_\epsilon^{conv}$: the set \mathcal{T}^{conv} is the set of all multivalued mappings of $\mathcal{T}^{u.l.s.}$ with convex (possibly empty) values; the set $\mathcal{T}_\epsilon^{conv}$ is defined as follows:

Definition 2.8 *The set $\mathcal{T}_\epsilon^{conv}$ is the set of all multivalued mappings of $\mathcal{T}^{u.l.s.}$ such that for every $\lambda \in \mathbb{R}$, one has $coT_\lambda \subset T_{\lambda-\epsilon}$.*

We will see in the next section the interpretation of this last condition in terms of distance, which is roughly that the multivalued mapping T and its convex hull coT are not too far. Throughout this paper, all the subsets of \mathcal{F} and of \mathcal{T} are equipped with the topologies induced by Top' and Top .

We now state the following refinement of Theorem 2.2:

Theorem 2.9 *i) The mapping $\Phi : (\mathcal{F}, \|\cdot\|_\infty) \rightarrow (\mathcal{T}^{u.l.s.}, \delta)$ defined by*

$$\forall f \in \mathcal{F}, \forall \lambda \in \mathbb{R}, \Phi(f)(\lambda) = S_\lambda(f) := \{x \in X, f(x) \geq \lambda\}$$

is an isometric isomorphism, i.e. for every $(f, g) \in \mathcal{F} \times \mathcal{F}$, one has $\|f - g\|_\infty = \delta(\Phi(f), \Phi(g))$.

ii) Φ is an isometric isomorphism from \mathcal{F}^q to \mathcal{T}^{conv} , and also from \mathcal{F}_ϵ^q to $\mathcal{T}_\epsilon^{conv}$.

Proof. See the appendix.

3 The Convex hull operator

In this section, one defines the convex hull operator co from the set of all multivalued mapping from \mathbb{R} to X to itself as follows: for every multivalued mapping T from \mathbb{R} to X and for every $\lambda \in \mathbb{R}$, $(\text{co}T)_\lambda$ is the convex hull of T_λ .

We have seen in the introduction that a natural method to build the quasi-concave envelop of a mapping f is to take the convex hull $\text{co}\Phi(f)$ of the upper level sets of f , then to define a new quasi-concave mapping $\Phi^{-1}(\text{co}\Phi(f))$, if it is possible. A problem is that the multivalued mapping $\text{co}\Phi(f)$ may not be in $\mathcal{T}^{u.l.s.}$, so that $\Phi^{-1}(\text{co}\Phi(f))$ could be not well defined. The following example provides a u.l.s. multivalued mapping T from \mathbb{R} to \mathbb{R} whose convex hull is not u.l.s.

Example 1 Let T be the multivalued mapping from \mathbb{R} to \mathbb{R} defined by

$$\begin{aligned} T_x &= [x, -x] \cup ([0, 1[\cap \mathbb{Q}) \text{ if } x \leq 0 \\ T_x &= ([0, 1 - x[\cup [x, 1] \cap \mathbb{Q}) \text{ if } x \in [0, 1[\\ T_1 &= \{0\} \\ T_x &= \emptyset \text{ if } x > 1 \end{aligned}$$

It is easy to check that $T \in \mathcal{T}^{u.l.s.}$. Yet, the multivalued mapping $\text{co}T$ is defined by

$$\begin{aligned} \text{co}T_x &= [x, -x] \text{ if } x \leq -1 \\ \text{co}T_x &= [x, 1[\text{ if } x \in]-1, 0] \end{aligned}$$

$$coT_x = [0, 1[\text{ if } x \in [0, 1[$$

$$coT_1 = \{0\}$$

$$coT_x = \emptyset \text{ if } x > 1$$

and clearly, coT does not satisfy Property ii) of Definition 2.1 at $x = 1$ (left continuity), so is not u.l.s.

Example 2 Let f be the mapping from \mathbb{R}^2 to \mathbb{R} defined as follows: for every $(x, y) \in \mathbb{R}^2$, $f(x, y)$ is the opposite of the Euclidean distance between (x, y) and the set

$$K = \{(0, 0)\} \cup \{(x, \frac{1}{x}) \in \mathbb{R}^2, x < 0\}.$$

The mapping f is continuous, $\Phi(f)(0) = K$ and $co\Phi(f)(0) = coK$ is not closed. Now, for every integer $n > 0$, one has

$$\overline{coK} \subset co\Phi(f)(-\frac{1}{n}),$$

so that

$$\bigcap_{n \in \mathbb{N}^*} co\Phi(f)(-\frac{1}{n}) \neq co\Phi(f)(0),$$

and so the multivalued function $co\Phi(f)$ is not u.l.s.

So, the first task to avoid this problem is to extend $\Psi = \Phi^{-1}$ on a larger set. We prove that Ψ is, as a matter of fact, an isometry on all the set \mathcal{T} :

Proposition 3.1

$$\forall T \in \mathcal{T}, \Psi(T)(x) = \sup\{\lambda \in \mathbb{R} \mid x \in T_\lambda\}$$

is well defined, and Ψ is an isometry. More precisely, for every $(T, T') \in \mathcal{T} \times \mathcal{T}$, one has $\|\Psi(T) - \Psi(T')\|_\infty = \delta(T, T')$.

Proof. See the appendix.

Unfortunately, there could remain some multivalued mapping $T \in \mathcal{T}^{u.l.s.}$ such that coT is not in \mathcal{T} , as shown in the following example:

Example 3 Let T be the multivalued mapping from \mathbb{R} to \mathbb{R} defined by

$$T_x =]-\infty, -x] \cup [x, +\infty[\text{ if } x \geq 0$$

$$T_x = \mathbb{R} \text{ if } x \leq 0$$

It is clear that $T \in \mathcal{T}^{u.l.s.}$, but $coT \notin \mathcal{T}$, because $\bigcap_{x \in \mathbb{R}} coT_x = \mathbb{R}$.

So, the second task is to find a subset \mathcal{T}' of \mathcal{T} such that for every $T \in \mathcal{T}'$, $coT \in \mathcal{T}$. This is the aim of the following definition:

Definition 3.2 Define \mathcal{T}' the set of multivalued mappings of \mathcal{T} such that

$$\bigcap_{\lambda \in \mathbb{R}} coT_\lambda = \emptyset.$$

It is clear from the previous definition that $co(\mathcal{T}') \subset \mathcal{T}$. Remark that for every bounded mapping $f : X \rightarrow \mathbb{R}$, $\Phi(f) \in \mathcal{T}'$, because $\Phi(f)(n) = \emptyset$ for $n \in \mathbb{N}$ large enough.

The following proposition relates this new set \mathcal{T}' to $\mathcal{T}_\epsilon^{conv}$:

Proposition 3.3 For every $\epsilon > 0$, $\mathcal{T}_\epsilon^{conv} \subset \mathcal{T}'$.

Proof of Proposition 3.3 For every $\epsilon > 0$, let $T \in \mathcal{T}_\epsilon^{conv}$. One has $\bigcap_{\lambda \in \mathbb{R}} coT_\lambda \subset \bigcap_{\lambda \in \mathbb{R}} T_{\lambda-\epsilon}$ (from the definition of $\mathcal{T}_\epsilon^{conv}$). This last set is also $\bigcap_{\lambda \in \mathbb{R}} T_\lambda = \emptyset$ from Property iii) of the definition of $\mathcal{T}^{u.l.s.}$.

To finish, notice that each point of $\mathcal{T}_\epsilon^{conv}$ is almost stabilized by the convex hull operator:

Proposition 3.4 For every $T \in \mathcal{T}_\epsilon^{conv}$, $\delta(co(T), T) \leq \epsilon$.

Proof. This is clear from the definition of $\mathcal{T}_\epsilon^{conv}$.

4 Stability of quasi-concavity

To illustrate the utility of the previous results, one proves the stability of the quasi-concave property.

Let $f \in \mathcal{F}_\epsilon^q$ an ϵ -quasi-concave mapping from X to \mathbb{R} . By Theorem 2.9, $\Phi(f) \in \mathcal{T}_\epsilon^{conv}$, so $\Phi(f) \in \mathcal{T}'$ from Proposition 3.3. Since $co(\mathcal{T}') \subset \mathcal{T}$, one has $co(\Phi(f)) \in \mathcal{T}$. Hence, from Proposition 3.1, $\tilde{f} := \Psi(co(\Phi(f)))$ is well defined. Besides, since $\Psi(\Phi(f)) = f$, one has $\|\tilde{f} - f\|_\infty = \|\Psi(co(\Phi(f))) - \Psi(\Phi(f))\|_\infty$. Now, from Proposition 3.1, Ψ is an isometry on \mathcal{T} ; thus, one has $\|\Psi(co(\Phi(f))) - \Psi(\Phi(f))\|_\infty = \delta(co(\Phi(f)), \Phi(f)) \leq \epsilon$ from Proposition 3.4 applied to $T = \Phi(f) \in \mathcal{T}_\epsilon^{conv}$.

Thus, one obtains a result similar to Hyers-Ulam's one for quasi-concave mappings:

Proposition 4.1 *For every ϵ -quasi-concave mapping $f : X \rightarrow \mathbb{R}$ there exists $\tilde{f} : X \rightarrow \mathbb{R}$ a quasi-concave mapping such that $\|\tilde{f} - f\|_\infty \leq \epsilon$.*

5 Appendix

Proof of Proposition 2.4. First notice that by definition, $\delta(T, T')$ is positive for every $(T, T') \in \mathcal{T} \times \mathcal{T}$, and may be infinite. Besides, $\delta(T, T) = 0$ for every $T \in \mathcal{T}$, and δ is clearly symmetric. To prove the triangular inequality, let $(T, T', T'') \in \mathcal{T}^3$ such that $\delta(T, T') < +\infty$ and $\delta(T', T'') < +\infty$ (if one of these two inequalities are not true, then the inequality $\delta(T, T'') \leq \delta(T, T') + \delta(T', T'')$ is clearly true). First, from the definition of $\delta(T, T')$ and from the monotony assumption satisfied by T and T' , one has

$$\forall \lambda \in \mathbb{R}, \forall \epsilon > 0, T_\lambda \subset T'_{\lambda - \delta(T, T') - \epsilon}. \quad (1)$$

Second, one can apply Equation 1 to T' and T'' , replacing λ by $\lambda - \delta(T, T') - \epsilon$. One obtains

$$\forall \lambda \in \mathbb{R}, \forall \epsilon > 0, T'_{\lambda - \delta(T, T') - \epsilon} \subset T''_{\lambda - \delta(T, T') - \delta(T', T'') - 2\epsilon}. \quad (2)$$

Combination of Equations 1 and 2 gives, for every $\lambda \in \mathbb{R}$ and every $\epsilon > 0$,

$$T_\lambda \subset T''_{\lambda - \delta(T, T') - \delta(T', T'') - 2\epsilon}$$

and similarly, by symmetry,

$$T''_\lambda \subset T_{\lambda - \delta(T, T') - \delta(T', T'') - 2\epsilon}.$$

Thus, from the definition of $\delta(T, T'')$, for every $\epsilon > 0$ one has

$$\delta(T, T'') \leq \delta(T, T') + \delta(T', T'') + 2\epsilon$$

which entails the triangle inequality.

Thus, δ is an generalized pseudo metric, that is to say a pseudo metric that can take infinite values.

Besides, if $\delta(T, T') = 0$ for some $(T, T') \in \mathcal{T}^{u.l.s.} \times \mathcal{T}^{u.l.s.}$, then for every $\lambda \in \mathbb{R}$, one has

$$T_\lambda \subset \bigcap_{n \in \mathbb{N}} T'_{\lambda - \frac{1}{n}}.$$

But from Property ii) of Definition 2.1, this last set is equal to T'_λ ; similarly, one obtains, for every $\lambda \in \mathbb{R}$, $T'_\lambda \subset T_\lambda$, and finally $T = T'$. Thus, the restriction of δ to $\mathcal{T}^{u.l.s.} \times \mathcal{T}^{u.l.s.}$ is a generalized metric.

Proof of Theorem 2.9

Statement i.) One has to prove that for every $(f, g) \in \mathcal{F} \times \mathcal{F}$, $\|f - g\|_\infty = \delta(\Phi(f), \Phi(g))$. Let $(f, g) \in \mathcal{F} \times \mathcal{F}$. Define $\epsilon := \|f - g\|_\infty \in \mathbb{R} \cup \{+\infty\}$

First case: ϵ finite. We want to prove that $\delta(\Phi(g), \Phi(f)) = \epsilon$.

First, prove the inequality $\delta(\Phi(g), \Phi(f)) \leq \epsilon$. This would be a consequence of the fact that for every $\lambda \in \mathbb{R}$,

$$\Phi(g)(\lambda) \subset \Phi(f)(\lambda - \epsilon)$$

and

$$\Phi(f)(\lambda) \subset \Phi(g)(\lambda - \epsilon).$$

To prove the first inclusion, let $x \in \Phi(g)(\lambda)$, which means $g(x) \geq \lambda$. From $\|f - g\|_\infty = \epsilon$, one deduces that $f(x) \geq \lambda - \epsilon$, thus $x \in \Phi(f)(\lambda - \epsilon)$. The second inclusion is proved similarly.

Now, prove the converse inequality $\delta(\Phi(g), \Phi(f)) \geq \epsilon$. If this inequality is false, then there exists $a > 0$ such that

$$\delta(\Phi(g), \Phi(f)) < \epsilon - a.$$

From the definition of Ψ , for every $x \in X$ there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ converging to $\Psi(\Phi(f))(x) = f(x)$ such that $x \in \Phi(f)(\lambda_n)$ for every integer n . From $\delta(\Phi(f), \Phi(g)) < \epsilon - a$, the condition $x \in \Phi(f)(\lambda_n)$ implies

$$x \in \Phi(g)(\lambda_n - \epsilon + a),$$

which is equivalent by definition to

$$g(x) \geq \lambda_n - \epsilon + a.$$

Passing to the limit, one obtains $g(x) \geq f(x) - \epsilon + a$, and similarly, one proves $f(x) \geq g(x) - \epsilon + a$. Thus, $\|f(x) - g(x)\|_\infty < \epsilon$, a contradiction with the definition of ϵ .

Second case: $\epsilon = +\infty$. We want to prove that $\delta(\Phi(g), \Phi(f)) = +\infty$; suppose that

$$\delta(\Phi(g), \Phi(f)) < A$$

for some $A \in \mathbb{R}$. From the definition of Ψ , for every $x \in X$, there exists a sequence $(\lambda_n)_{n \in \mathbb{N}}$ converging to $\Psi(\Phi(f))(x) = f(x)$ such that $x \in \Phi(f)(\lambda_n)$ for every integer n . From $\delta(\Phi(f), \Phi(g)) < A$, the condition $x \in \Phi(f)(\lambda_n)$ implies

$$x \in \Phi(g)(\lambda_n - A),$$

or equivalently, for every integer n ,

$$g(x) \geq \lambda_n - A.$$

Passing to the limit, one obtains $g(x) \geq f(x) - A$, and similarly, one proves $f(x) \geq g(x) - A$. Thus, $\|f(x) - g(x)\|_\infty \leq A$, a contradiction with the definition of ϵ .

Statement ii.) It is well known that f is quasi-concave if and only if its upper level sets are convex; thus, Φ is an isometric isomorphism from \mathcal{F}^q to \mathcal{T}^{conv} . Now, let us prove that f is ϵ -quasi-concave if and only if $\Phi(f) \in \mathcal{T}_\epsilon^{conv}$. First suppose that f is ϵ -quasi-concave. Let $\lambda \in \mathbb{R}$ and $x \in \text{co}S_\lambda(f)$. Thus, there exists an integer n and x_1, \dots, x_n in $S_\lambda(f)$ such that $x = \sum_{i=1}^n \mu_i x_i$, with $\mu_i \geq 0$ for every $i = 1, \dots, n$ and $\sum_{i=1}^n \mu_i = 1$. From the definition of $S_\lambda(f)$, for every $i = 1, \dots, n$ one has $f(x_i) \geq \lambda$; since f is ϵ -quasi-concave, one obtains $f(x) \geq \lambda - \epsilon$, or equivalently $x \in S_{\lambda-\epsilon}(f)$. We have proved that for every $\lambda \in \mathbb{R}$, one has $\text{co}S_\lambda(f) \subset S_{\lambda-\epsilon}(f)$, or equivalently, $\Phi(f) \in \mathcal{T}_\epsilon^{conv}$.

Conversely, suppose $\text{co}S_\lambda(f) \subset S_{\lambda-\epsilon}(f)$ for every $\lambda \in \mathbb{R}$. Let $x = \sum_{i=1}^n \mu_i x_i$, with $\mu_i \geq 0$ for every $i = 1, \dots, n$ and $\sum_{i=1}^n \mu_i = 1$. Define

$$\lambda = \min\{f(x_1), \dots, f(x_n)\},$$

so that one has $f(x_i) \geq \lambda$ for every $i = 1, \dots, n$; equivalently, this means $x_i \in S_\lambda(f)$ for every $i = 1, \dots, n$. Consequently, one has $x \in \text{co}S_\lambda(f)$. From

$\text{co}S_\lambda(f) \subset S_{\lambda-\epsilon}(f)$, one finally obtains $x \in S_{\lambda-\epsilon}(f)$, which is equivalent to $f(x) \geq \min\{f(x_1), \dots, f(x_n)\} - \epsilon$. This proves that f is ϵ -quasi-concave. Finally, Φ is an isometric isomorphism from \mathcal{F}_ϵ^q to $\mathcal{T}_\epsilon^{\text{conv}}$.

Proof of Proposition 3.1

Notice that Ψ is clearly well defined on \mathcal{T} , and not only on $\mathcal{T}^{u.l.s.}$.

First prove that

$$\delta(T, T') \leq \|\Psi(T) - \Psi(T')\|_\infty.$$

By definition of $\delta(T, T')$, there exist two sequences $(\lambda_n)_{n \in \mathbb{N}}$ and $(\mu_n)_{n \in \mathbb{N}}$ in \mathbb{R} , where $(\mu_n)_{n \in \mathbb{N}}$ converges to $\delta(T, T')$, and such that for every integer n , T_{λ_n} is not contains in $T'_{\lambda_n - \mu_n}$. It means that for every integer n , there exists $x_n \in X$ such that $x_n \in T_{\lambda_n}$ and $x_n \notin T'_{\lambda_n - \mu_n}$. Thus, by definition, for every integer n one has

$$\Psi(T)(x_n) \geq \lambda_n$$

and

$$\Psi(T')(x_n) \leq \lambda_n - \mu_n.$$

Thus,

$$\|\Psi(T) - \Psi(T')\|_\infty \geq \mu_n,$$

and one obtains at the limit $\|\Psi(T) - \Psi(T')\|_\infty \geq \delta(T, T')$. Notice that this proof works if $\delta(T, T') = +\infty$.

Now prove that

$$\delta(T, T') \geq \|\Psi(T) - \Psi(T')\|_\infty.$$

Suppose this last inequality is false, i.e. $\delta(T, T') < \|\Psi(T) - \Psi(T')\|_\infty$. Thus, there exists $x \in X$ such that for $\epsilon > 0$ small enough, one has

$$\Psi(T)(x) - \Psi(T')(x) > \delta(T, T') + \epsilon$$

(switching T and T' is necessary). Thus, from the definition of $\Psi(T')(x)$, one has

$$x \notin T'_{\Psi(T)(x) - \delta(T, T') - \epsilon}.$$

But from the definition of $\Psi(T)(x)$, one also have

$$x \in T_{\Psi(T)(x) - \frac{\epsilon}{2}}.$$

From the definition of $\delta(T, T')$, it implies $\delta(T, T') \geq (\Psi(T)(x) - \frac{\epsilon}{2}) - (\Psi(T')(x) - \delta(T, T') - \epsilon) = \delta(T, T') + \frac{\epsilon}{2}$, a contradiction.

References

- [1] D. H. Hyers, S. M. Ulam, "Approximately Convex Functions", Proceedings of the A.M.S., Vol. 3, No. 5. (1952), pp. 821-828.
- [2] Hadjisavvas, N., Komlsi, S., Schaible, S. (2005) "Handbook of Generalized Convexity and Generalized Monotonicity" , Nonconvex Optimization and its Applications, 76.